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*Published in:*  
Physical Review A

*DOI:*  
[10.1103/PhysRevA.93.032107](https://doi.org/10.1103/PhysRevA.93.032107)

*Publication date:*  
2016

*Citation for published version (APA):*  
Kiukas, J., & Burgarth, D. (2016). Quantum resource control for noisy Einstein-Podolsky-Rosen steering with qubit measurements. *Physical Review A*, 93(3), [032107]. <https://doi.org/10.1103/PhysRevA.93.032107>

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# Quantum resource control for noisy Einstein-Podolsky-Rosen steering with qubit measurements

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(Received 16 October 2015; published 4 March 2016)

We demonstrate how quantum optimal control can be used to enhance quantum resources for bipartite one-way protocols, specifically Einstein-Podolsky-Rosen steering with qubit measurements. Steering is relevant for one-sided device-independent key distribution, the realistic implementations of which necessitate the study of noisy scenarios. So far, mainly the case of imperfect detection efficiency has been considered; here we look at the effect of dynamical noise responsible for decoherence and dissipation. In order to set up the optimization, we map the steering problem into the equivalent joint measurability problem and employ quantum resource-theoretic robustness monotones from that context. The advantage is that incompatibility (hence steerability) with arbitrary pairs of noisy qubit measurements has been completely characterized through an analytical expression, which can be turned into a computable cost function with exact gradient. Furthermore, dynamical loss of incompatibility has recently been illustrated by using these monotones. We demonstrate resource control numerically by using a special gradient-based software showing, in particular, the advantage over naive control with cost function chosen as a fidelity in relation to a specific target. We subsequently illustrate the complexity of the control landscapes with a simplified two-variable scheme. The results contribute to the theoretical understanding of the limitations in realistic implementations of quantum information protocols, also paving the way to practical use of the rather abstract quantum resource theories.

DOI: [10.1103/PhysRevA.93.032107](https://doi.org/10.1103/PhysRevA.93.032107)

## I. INTRODUCTION

Due to the emerging technological motivation, it has become popular to view quantum effects as *resources* for tasks which cannot be described by using classical physics [1–3]. While most work focuses on nonclassical properties of quantum states, measurement resources are just as important, since the set of available measurements in any real experiment is restricted by the implementable *controls* such as laser pulses [4]. This is particularly relevant in correlation experiments where local parties make measurements on a shared entangled state. If the correlations violate a Bell inequality, they can be used in quantum key distribution without any knowledge of the measurement devices; however, this is experimentally difficult due to the detection loophole [5,6]. Implementation is less challenging in a semi-device-independent scenario based on Einstein-Podolsky-Rosen (EPR) steering [7] (Fig. 1), which has attracted considerable interest recently [8–16]. It is intriguing because it requires entanglement but can be done with correlations admitting a hidden-variable model. For instance, steering is possible with Gaussian states and measurements [7] which cannot violate Bell inequalities due to the hidden-variable model provided by the Wigner function.

The intuitive idea is Alice “steering” Bob with her measurements through the shared state, which “transmits” an “assemblage” of conditional states to Bob [7,8]. The maximally entangled state provides perfect transmission, and the general case reduces to that by replacing Alice’s measurements by certain state-dependent ones Bob reconstructs from the assemblage [16]. Hence, the quantum resource for steering can be described entirely by measurements; this is very useful when treating the loss of steerability due to local noise, as we see below. Interestingly, the required measurement resource has an independent meaning [14–16]: the measurements need to be *incompatible*. Here, incompatibility does not mean

noncommutativity, because noisy measurements are typically not projective. It means the nonexistence of a “hidden” measurement jointly simulating all of Alice’s measurements; this notion has been studied for a long time [17–25].

Hence, the loss of steerability can be described independently of the bipartite scenario, as the loss of incompatibility on Bob’s side. This leads to a simplification in system size, the simplest case being a single qubit. In order to *quantify* this loss, we need a numerical *incompatibility monotone*; they can be constructed [16,26] by using the noise-robustness idea from general quantum resource theories [1,27–31]. The dynamics of incompatibility has recently been studied by using these monotones [32].

In this paper, we take another direction by showing how steering resource can be *directly* enhanced in the presence of Markovian noise, using numerical gradient-based quantum optimal control [4,33,34]. Research in this area aims at characterizing operations reachable with a restricted set of controls, such as laser pulses, and numerically finding optimal pulses implementing a given target. While unitary control is fairly well established, control of noisy operations (quantum channels) is more challenging due to their complicated structure even in small systems [35–38]. In contrast to the usual optimization of a distance from a specific target, we optimize over an incompatibility monotone, so as to do *purpose-oriented control* of EPR steering, in analogy to entanglement control [39–41]. Steering is more challenging in small systems, where the existence of a suitable hidden-variable model is already a nontrivial question; we use the special characterization of qubit incompatibility [24,25] to compute a faithful incompatibility monotone with exact gradient. The purpose-oriented control of steering rather than targeting specific measurements is motivated by the fact that *many* measurements have equal steering potential, and that

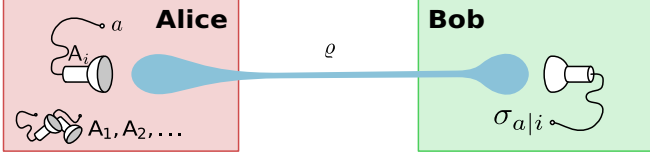


FIG. 1. A correlation experiment for EPR steering.

targeting specific ideal (projective) ones is not likely to work as they are usually not reachable in noisy systems. The problem is also intriguing in that the monotones are unitary-invariant; unitary control *can only help in the presence of noise* which destroys the resource in the first place; hence it is *a priori* not at all clear if it actually does help.

## II. THE QUANTUM RESOURCE FOR STEERING

We look at the bipartite scenario with Alice and Bob sharing a state  $\rho$  on the tensor product Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$ . Alice has a restricted set  $\mathcal{M}_A = \{A_1, \dots, A_n\}$  of measurements, assumed to be general (possibly nonprojective) positive-operator-value measures (POVMs), as this is necessary in noisy scenarios. Hence each  $A_i$  stands for a collection of positive-semidefinite matrices  $A_i(a)$  with  $\sum_a A_i(a) = \mathbb{1}$ . The steering protocol proceeds as follows [7]: Alice chooses index  $i$ , performs  $A_i$  on her system getting an outcome  $a$ , and announces  $(i, a)$  to Bob, who uses state tomography to extract the assemblage [9]  $\sigma_{a|i} := \text{tr}_A\{\rho[A_i(a) \otimes \mathbb{1}]\}$  of conditional states. The assemblage is called steerable if Bob cannot reconstruct it from some pre-existing collection of hidden states by using only classical information on Alice's results. We state the precise definition in terms of the associated joint measurability problem [14–16]: following Ref. [16] we restrict without loss of generality  $\mathcal{H}_B = \text{ran } \rho_B$  and define

$$S_\rho(A) := \rho_B^{-\frac{1}{2}} \text{tr}_A[\rho(A \otimes \mathbb{1})]^\top \rho_B^{-\frac{1}{2}},$$

so that  $S_\rho$  is a unital completely positive map between Alice's and Bob's observable algebras, and  $\text{tr}\{\rho[\mathbb{1} \otimes S_\rho(A)]\} = \text{tr}\{\rho(A \otimes \mathbb{1})\}$  for any matrix  $A$ . Here  $B^\top$  stands for the transpose of a matrix  $B$ . This means that Bob can simulate Alice's measurements via the POVMs  $B_i^\rho = S_\rho(A_i)$  determined by the assemblage. The setting is *nonsteerable* if these are *jointly measurable* in that each  $B_i^\rho$  is a classical probabilistic postprocessing of a single POVM  $G(\lambda)$ ; formally,

$$B_i^\rho(a) = \sum_\lambda p(a|i, \lambda) G(\lambda) \text{ for all } i = 1, \dots, n,$$

where  $\sum_a p(a|i, \lambda) = 1$ . This definition is equivalent to the usual notion of joint measurability via marginals [13].

In conclusion, the quantum resource for EPR steering can be characterized as the opposite of joint measurability, often called *incompatibility*, of the collection

$$B_i^\rho = S_\rho(A_i), \quad i = 1, \dots, n$$

of measurements. This formulation has the advantage of referring only to a single system; the “nonlocal” aspect is encapsulated in  $S_\rho$ . This is especially useful in describing local noise: given a channel  $\Lambda$  on Alice's side changing the

state as  $\rho \mapsto \tilde{\rho} = (\Lambda \otimes \text{Id})(\rho)$ , the assemblage changes

$$\sigma_{a|i} \mapsto \tilde{\sigma}_{a|i} = \text{tr}_A(\rho\{\Lambda^*[A_i(a)] \otimes \mathbb{1}\}).$$

Then the resource transforms into

$$\tilde{B}_i^\rho = S_\rho \circ \Lambda^*(A_i), \quad i = 1, \dots, n, \quad (1)$$

the Heisenberg channel  $\Lambda^*$  simply concatenating with  $S_\rho$ . This has a clear interpretation:  $S_\rho$  describes *preparation noise* in an imperfect production of a maximally entangled state (for which  $S_\rho = \text{Id}$ ), while  $\Lambda$  is the subsequent *dynamical noise*. Steerability of the noisy assemblage is equivalent to the incompatibility of Eq. (1).

Resource theories also contain the idea of *quantification* [1]. Incompatibility of a collection  $(B_1, \dots, B_n)$  of measurements can be quantified by an *incompatibility monotone* [26], i.e., a number  $\mathcal{I}(B_1, \dots, B_n)$  which is zero exactly in the jointly measurable case, and

$$\mathcal{I}(\Lambda^*(B_1), \dots, \Lambda^*(B_n)) \leq \mathcal{I}(B_1, \dots, B_n) \quad (2)$$

for any positive linear map  $\Lambda$ . This fits well with the steering resource (1), where the total channel  $S_\rho \circ \Lambda^*$  then effects a *quantitative* loss of the resource. In particular, loss due to continuous dynamics  $t \mapsto \Lambda_t$  is described by the function  $t \mapsto \mathcal{I}(S_\rho \circ \Lambda_t^*(A_1), \dots, S_\rho \circ \Lambda_t^*(A_n))$ .

Good operational incompatibility monotones describe convex-geometric *noise robustness* [1, 16, 26, 31]: we mix *classical* noise with distribution  $p = (p_i)$  into measurements via  $A \mapsto (1 - \lambda)A + \lambda p \mathbb{1}$  and define  $\mathcal{I}$  to be the minimal  $\lambda$  for which the resource is lost, i.e., measurements become jointly measurable (see Refs. [26, 32] for discussion).

In this paper, we look at the simplest setting with  $\mathcal{H}_A = \mathbb{C}^2$  and two measurements for Alice; this case is already interesting. The speciality is that robustness monotones can be computed by using the analytical characterization of incompatibility [24, 25]: Given a POVM  $A = (A, \mathbb{1} - A)$  on  $\mathbb{C}^2$ , we identify  $A = \frac{1}{2}(x^0 \mathbb{1} + \mathbf{x} \cdot \boldsymbol{\sigma})$  with the four-vector  $x = (x^0, \mathbf{x})$ , and  $\mathbb{1} - A$  with  $x^\perp := (2 - x^0, -\mathbf{x})$ . The condition  $0 \leq A \leq \mathbb{1}$  reads  $x, x^\perp \in \mathcal{F}_+$ , where  $\mathcal{F}_+ = \{x \mid \langle x|x \rangle \geq 0, x^0 \geq 0\}$  and  $\langle x|y \rangle := x^0 y^0 - \sum_{i=1}^3 x^i y^i$  is the Minkowski form. A pair of measurements  $x_1$  and  $x_2$  is jointly measurable *if and only if*  $\mathcal{C}(x, y) \geq 0$ , where

$$\begin{aligned} \mathcal{C}(x_1, x_2) := & [\langle x_1|x_1 \rangle \langle x_1^\perp|x_1^\perp \rangle \langle x_2|x_2 \rangle \langle x_2^\perp|x_2^\perp \rangle]^{1/2} \\ & - \langle x_1|x_1^\perp \rangle \langle x_2|x_2^\perp \rangle + \langle x_1|x_2^\perp \rangle \langle x_1^\perp|x_2 \rangle \\ & + \langle x_1|x_2 \rangle \langle x_1^\perp|x_2^\perp \rangle. \end{aligned}$$

We use the robustness monotone  $\mathcal{I}_b(x, y)$  of Ref. [26]: given a probability  $p = \frac{1}{2}(1 + b)$ , the above classical noise is  $x \mapsto N_{\lambda, b}(x) := [(1 - \lambda)x^0 + \lambda 2p, (1 - \lambda)\mathbf{x}]$ , and  $\mathcal{I}_b(x, y)$  is by definition the unique solution  $0 \leq \lambda \leq 1/2$  of

$$\mathcal{C}(N_{\lambda, b}(x_1), N_{\lambda, b}(x_2)) = 0. \quad (3)$$

Interestingly,  $\mathcal{I}_0(x_1, x_2)$  coincides with the maximal violation of the Clauser-Horne-Shimony-Holt-Bell (CHSH-Bell) inequality with Alice's measurements  $(x_1, x_2)$  [26]. This monotone was recently used in studying the loss of incompatibility on open systems [32].

### III. OPTIMAL RESOURCE CONTROL OF STEERING

Having identified and quantified the resource for steering and described its loss in open systems, it is natural to ask if this loss can be slowed down by control. As discussed in the introduction, our approach is to directly optimize the incompatibility resource (1) by using the monotones  $\mathcal{I}_b$ .

We take Alice's noise to be Markovian:  $\Lambda_t = e^{t\mathcal{L}_0}$  with a drift Lindbladian  $\mathcal{L}_0$ . We look at two basic cases: amplitude damping  $\mathcal{L}_0^{\text{AD}}(\rho) = \gamma(2\sigma_- \rho \sigma_+ - \{\sigma_+ \sigma_-, \rho\})$  (containing decoherence and dissipation), and dephasing in the  $\sigma_y$  basis  $\mathcal{L}_0^{\text{DP}}(\rho) = \gamma(\sigma_y \rho \sigma_y^\dagger - \rho)$  (only decoherence), with  $\gamma = 0.1$ . We model control (e.g., a laser pulse) by changing  $\mathcal{L}_0$  into

$$\mathcal{L}_c = \mathcal{L}_0 - ic[H, \cdot],$$

where the  $H$  is a control Hamiltonian and  $c \in \mathbb{R}$ . By applying a sequence  $\mathbf{c} = (c_1, \dots, c_m)$  of such pulses, each of duration  $\Delta t$ , the dynamics at time  $T = m\Delta t$  is  $\Lambda_c = e^{\Delta t \mathcal{L}_{c_m}} \dots e^{\Delta t \mathcal{L}_{c_1}}$ . We consider the setting where Alice aims at steering Bob at time  $T$  with a measurement pair  $(x_1, x_2)$ , given that the initial state

was  $\rho$ . According to the preceding section, the information on the resource is faithfully encoded into the cost function

$$f(\mathbf{c}) = \mathcal{I}_b(S_\rho \circ \Lambda_c^*(x_1), S_\rho \circ \Lambda_c^*(x_2)) \quad (4)$$

describing the steering robustness. In particular, this function is nonzero if and only if the setting is steerable. We implemented Eq. (4) numerically by solving Eq. (3) using standard root finding, which needs only a few iterations because the function is just a combination of polynomials and square roots and changes sign on  $[0, 1/2]$ . Given this value, the gradient of  $\mathcal{I}_b$  can be found *analytically* via implicit differentiation. This fits particularly well with the control software QTRL [42], which implements the Fréchet derivative of  $\mathbf{c} \mapsto \Lambda_c$  to be used in computing cost functions; hence we get the gradient of  $f(\mathbf{c})$  from the chain rule and employ optimization based on the exact gradient.

We take a maximally entangled  $\rho$  so that  $S_\rho = \text{Id}$ , Alice's measurements  $(\sigma_x, \sigma_z)$ ,  $H = \sigma_y + \sigma_z$ , and use the monotone  $\mathcal{I}_0$ . The results for different times  $T$  are depicted in Fig. 2. It shows the robustness  $\mathcal{I}_0$  in the uncontrolled case, the optimized value, and comparison with the naive control strategy aiming at the channel closest to the identity. The computations were done with QTRL on HPC Wales with  $m = 20$  (number of time slots) and optimized over 100 random initial pulses. Optimization with the cost function (4) is considerably slower than the naive method; however, the results are significantly better. We also see that control works better with dephasing, presumably due to the lack of dissipation present in the amplitude damping.

An inspection of the optimal pulses showed that the amplitudes peak close to the end, suggesting that only a few time slots are needed. Accordingly, we considered the following simple scheme: drift until time  $t_{\text{drift}} < T$ , then apply two pulses  $c_1, c_2$ , each of duration  $\Delta t = (T - t_{\text{drift}})/2$ . This implements the map  $\Lambda_{c_1, c_2} = e^{\Delta t \mathcal{L}_{c_2}} e^{\Delta t \mathcal{L}_{c_1}} e^{t_{\text{drift}} \mathcal{L}_0}$ . The corresponding control landscape for  $f(c_1, c_2)$  in Eq. (4) is shown in Fig. 3 for the amplitude-damping case with  $t_{\text{drift}} = 2.6$  and  $T = 2.8$ . At this time resource control can

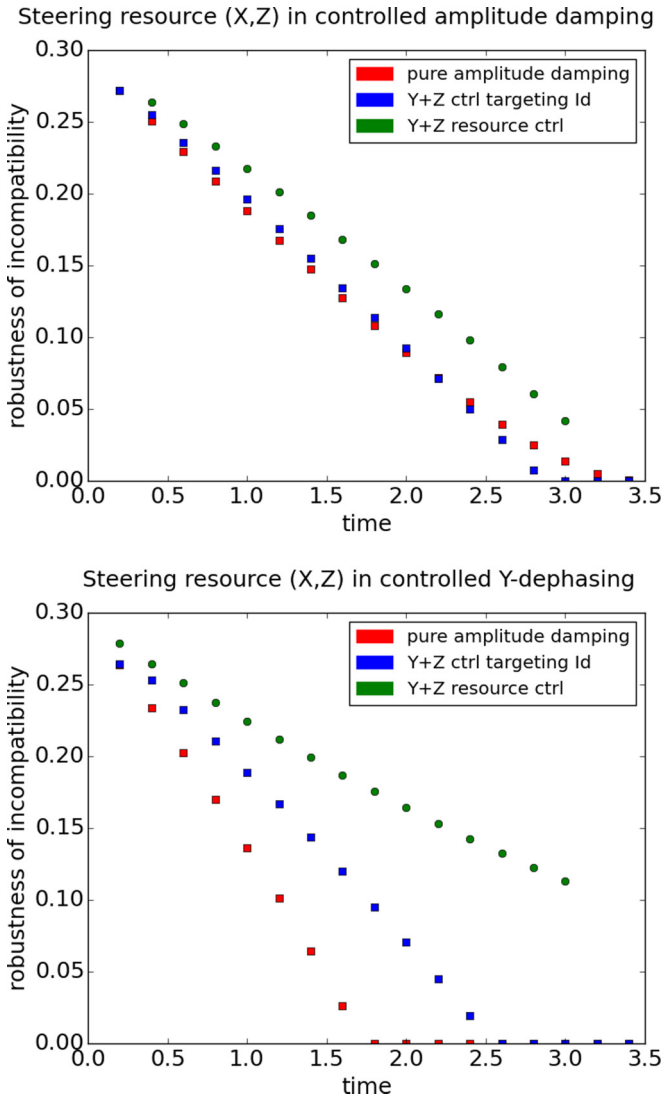


FIG. 2. Resource control of steering under amplitude damping (upper panel) and dephasing in the  $\sigma_y$  basis (lower panel).

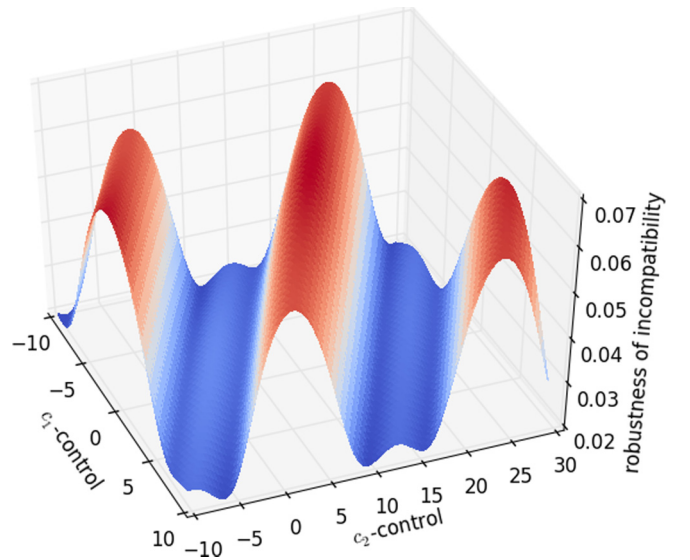


FIG. 3. A control landscape of steering robustness under amplitude-damping noise. The maximum represents optimal quantum resource control.



boost steering robustness up to 0.062, which is more than a factor-of-two improvement from the pure noise (see Fig. 2). The maximum is attained with the pulse  $\mathbf{c} = (-1.42, 12.32)$ . Clearly, the choice of controls is crucial; looking also at Fig. 2 we observe that both pure noise (no control) and the naive strategy are far from the maximum. In the dephasing case the landscape is similar, except that there are controls leading to nonsteerability (including the no control and naive control cases), while optimal control manages to preserve steering. The maximum steering robustness is 0.125 with the pulse  $\mathbf{c} = (1.80, -12.88)$ . It appears to be crucial that the controls are applied just before measuring, when incompatibility is close to its maximum.

#### IV. CONCLUSION AND OUTLOOK

We have demonstrated how steering can be enhanced by control in noisy qubit systems, with direct resource optimization of Eq. (4) performing better than the target-based one. The

effect of an imperfect initial state  $\rho$  and other non-Markovian features [32] remain to be studied. Our results pave the way for general schemes for implementing optimal noisy quantum resources in controlled open systems. The optimization naturally becomes slow in large systems, with analytical gradients no longer available. Nevertheless, Eq. (4) can always be computed efficiently via a semidefinite program [26], and approximations based on steering inequalities [10] could be used in analogy to the entanglement control [40] to make the computations feasible.

#### ACKNOWLEDGMENTS

We thank the QUTIP project [42] for providing the software (especially A. Pitchford for developing the pulse-optimization package), and HPC Wales for computing time. We acknowledge financial support from the EPSRC project EP/M01634X/1.

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